

A Class of Infinite Dimensional Diffusion Processes with Connection to Population Genetics

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Abstract

Starting from a sequence of independent Wright-Fisher diffusion processes on $[0, 1]$, we construct a class of reversible infinite dimensional diffusion processes on $\Delta_\infty := \{\mathbf{x} \in [0, 1]^\mathbb{N} : \sum_{i \geq 1} x_i = 1\}$ with GEM distribution as the reversible measure. Log-Sobolev inequalities are established for these diffusions, which lead to the exponential convergence to the corresponding reversible measures in the entropy. Extensions are made to a class of measure-valued processes over an abstract space S . This provides a reasonable alternative to the Fleming-Viot process which does not satisfy the log-Sobolev inequality when S is infinite as observed by W. Stannat [13].

Key words: Poisson-Dirichlet distribution, GEM distribution, Fleming-Viot process, log-Sobolev inequality.

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1 Introduction

Population genetics is concerned with the distribution and evolution of gene frequencies in a large population at a particular locus. The infinitely-many-neutral-alleles model describes the evolution of the gene frequencies under generation independent mutation, and resampling. In statistical equilibrium the distribution of gene frequencies is the well known Poisson-Dirichlet distribution introduced by Kingman [8]. When a sample of size n genes is selected from a Poisson-Dirichlet population, the distribution of the corresponding allelic partition is given explicitly by the *Ewens sampling formula*. This provides an important tool in testing neutrality of a population.

Let

$$\Delta_\infty = \{\mathbf{x} = (x_1, x_2, \dots) \in [0, 1]^\mathbb{N} : \sum_{k=1}^{\infty} x_k = 1\},$$

and

$$\nabla = \{\mathbf{x} = (x_1, x_2, \dots) \in [0, 1]^{\mathbb{N}} : x_1 \geq x_2 \geq \dots \geq 0, \sum_{k=1}^{\infty} x_k = 1\}.$$

The Poisson-Dirichlet distribution with parameter $\theta > 0$ (henceforth $PD(\theta)$) is a probability measure Π_θ on ∇ . We use $\mathbf{P}(\theta) = (P_1(\theta), P_2(\theta), \dots)$ to denote the ∇ -valued random variable with distribution Π_θ . The component $P_k(\theta)$ represents the proportion of the k th most frequent alleles. If u is the individual mutation rate and N is the effective population size, then the parameter $\theta = 4Nu$ is the population mutation rate. A different way of describing the distribution is through the following size-biased sampling. Let $U_k, k = 1, 2, \dots$, be a sequence of independent, identically distributed random variables with common distribution $Beta(1, \theta)$, and set

$$(1.1) \quad X_1^\theta = U_1, X_n^\theta = (1 - U_1) \cdots (1 - U_{n-1})U_n, n \geq 2.$$

Clearly $(X_1^\theta, X_2^\theta, \dots)$ is in space Δ_∞ . The law of $X_1^\theta, X_2^\theta, \dots$ is called the one parameter GEM distribution and is denoted by Π_θ^{gem} . The descending order of $X_1^\theta, X_2^\theta, \dots$ has distribution Π_θ . The sequence $X_k^\theta, k = 1, 2, \dots$ has the same distribution as the size-biased permutation of Π_θ .

Let $\xi_k, k = 1, \dots$ be a sequence of i.i.d. random variables with common diffusive distribution ν on $[0, 1]$, i.e., $\nu(x) = 0$ for every x in $[0, 1]$. Set

$$(1.2) \quad \Theta_{\theta, \nu} = \sum_{k=1}^{\infty} P_k(\theta) \delta_{\xi_k}.$$

It is known that the law of $\Theta_{\theta, \nu}$ is *Dirichlet* (θ, ν) distribution, and is the reversible distribution of the Fleming-Viot process with mutation operator (cf. [2])

$$(1.3) \quad Af(x) = \frac{\theta}{2} \int_0^1 (f(y) - f(x)) \nu(dx).$$

For $0 \leq \alpha < 1, \theta > -\alpha$, let $\{V_k : k = 1, 2, \dots\}$ be a sequence of independent random variables such that V_k is a $Beta(1 - \alpha, \theta + k\alpha)$ random variable for each k . Set

$$(1.4) \quad X_1^{\theta, \alpha} = V_1, X_n^{\theta, \alpha} = (1 - V_1) \cdots (1 - V_{n-1})V_n, n \geq 1.$$

The law of $X_1^{\theta, \alpha}, X_2^{\theta, \alpha}, \dots$ is called the two-parameter GEM distribution and is denoted by $\Pi_{\alpha, \theta}^{gem}$. The law of the descending order statistic of $X_1^{\theta, \alpha}, X_2^{\theta, \alpha}, \dots$ is called the two-parameter Poisson-Dirichlet distribution (henceforth $\Pi_{\alpha, \theta}$) studied thoroughly in Pitman and Yor [12]. The sequence $X_k^{\theta, \alpha}, k = 1, 2, \dots$ has the same distribution as the size-biased permutation of $\Pi_{\alpha, \theta}$. It is shown in Pitman [10] that the two-parameter Poisson-Dirichlet distribution is the most general distribution whose size-biased permutation has the same distribution as the GEM representation (1.4). A two-parameter ‘‘Ewens sampling formula’’ is obtained in [11]. Let $\Theta_{\theta, \alpha, \nu}$ be defined similarly to $\Theta_{\theta, \nu}$ with X_k^θ being replaced by $X_k^{\theta, \alpha}$. We call the law of $\Theta_{\theta, \alpha, \nu}$ a *Dirichlet* (θ, α, ν) distribution.

The Poisson-Dirichlet distribution and its two-parameter generalization have many similar structures including the urn construction in [7] and [3], GEM representation, sampling formula, etc.. But we have not seen a stochastic dynamic model similar to the infinitely-many-neutral-alleles model and the Fleming-Viot process developed for the two-parameter Poisson-Dirichlet distribution and $Dirichlet(\theta, \alpha, \nu)$ distribution.

As the first result in this paper, we are able to construct a class of reversible infinite dimensional diffusion processes, the GEM processes, so that both Π_θ^{gem} and its two-parameter generalization $\Pi_{\alpha, \theta}^{gem}$ appear as the reversible measures for appropriate parameters.

In [13], the log-Sobolev inequality is studied for the Fleming-Viot process with motion given by (1.3). It turns out that the log-Sobolev inequality holds only when the type space is finite. In the second result of this paper, we will first construct a measure-valued process that has the $Dirichlet(\theta, \nu)$ distribution as reversible measure. Then we will establish the log-Sobolev inequality for the process.

The rest of the paper is organized as follows. The GEM processes associated with Π_θ^{gem} and $\Pi_{\alpha, \theta}^{gem}$ are introduced in section 2. Section 3 includes the proof of uniqueness and the log-Sobolev inequality of the GEM process. Finally in section 4, the measure-valued process is introduced and the corresponding log-Sobolev inequality is established.

2 GEM Processes

For any $i \geq 1$, let a_i, b_i be two strictly positive numbers. We assume that

$$(2.1) \quad \inf_i b_i \geq \frac{1}{2}.$$

Let $X_i(t)$ be the unique strong solution of the stochastic differential equation

$$(2.2) \quad dX_i(t) = (a_i - (a_i + b_i)X_i(t))dt + \sqrt{X_i(t)(1 - X_i(t))}dB_i(t), X_i(0) \in [0, 1],$$

where $\{B_i(t) : i = 1, 2, \dots\}$ are independent one dimensional Brownian motions. It is known that the process $X_i(t)$ is reversible with reversible measure $\pi_{a_i, b_i} = Beta(2a_i, 2b_i)$. By direct calculation, the scale function of $X_i(\cdot)$ is given by

$$s_i(x) = \left(\frac{1}{4}\right)^{a_i + b_i} \int_{1/2}^x \frac{dy}{y^{2a_i}(1 - y)^{2b_i}}.$$

By (2.1), we have $\lim_{x \rightarrow 1} s_i(x) = +\infty$ for all i . Thus starting from the interior of $[0, 1]$, the process $X_i(t)$ will not hit the boundary 1 with probability one. Let $E = [0, 1)^\mathbb{N}$. The process

$$\mathbf{X}(t) = (X_1(t), X_2(t), \dots)$$

is then a E -valued Markov process. Consider the map

$$\Phi : E \rightarrow \bar{\Delta}_\infty, \quad \mathbf{x} = (x_1, x_2, \dots) \rightarrow (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots)$$

with

$$\varphi_1(\mathbf{x}) = x_1, \varphi_n(\mathbf{x}) = x_n(1 - x_1) \cdots (1 - x_{n-1}), n \geq 2.$$

Clearly Φ is a bijection and the process $\mathbf{Y}(t) = \Phi(\mathbf{X}(t))$ is thus a Markov process. Let $\bar{E} := [0, 1]^\mathbb{N}$ be the closure of E , $C(\bar{E})$ denote the set of all continuous function on \bar{E} , and $C_{cl}^2(\bar{E})$ be the set of functions in $C(\bar{E})$ that have second order continuous derivatives, and depend only on a finite number of coordinates. The sets $C(E)$ and $C_{cl}^2(E)$ will be the respective restrictions of $C(\bar{E})$ and $C_{cl}^2(\bar{E})$ on E . Then the generator of process $\mathbf{X}(t)$ is given by

$$Lf(\mathbf{x}) = \sum_{k=1}^{\infty} \left\{ x_k(1 - x_k) \frac{\partial^2 f}{\partial x_k^2} + (a_k - (a_k + b_k)x_k) \frac{\partial f}{\partial x_k} \right\}, \quad f \in C_{cl}^2(E),$$

and can be extended to $C_{cl}^2(\bar{E})$. The sets $B(E)$ and $B(\Delta_\infty)$ are bounded measurable functions on E and Δ_∞ , respectively.

Let $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_1, b_2, \dots)$, and

$$\mu_{\mathbf{a}, \mathbf{b}} = \prod_{k=1}^{\infty} \pi_{a_k, b_k}, \quad \Xi_{\mathbf{a}, \mathbf{b}} = \mu_{\mathbf{a}, \mathbf{b}} \circ \Phi^{-1}.$$

Then we have

Theorem 2.1 *The processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are reversible with respective reversible measures $\mu_{\mathbf{a}, \mathbf{b}}$ and $\Xi_{\mathbf{a}, \mathbf{b}}$.*

Proof: The reversibility of $\mathbf{X}(t)$ follows from the reversibility of each $X_i(t)$. Now for any two f, g in $B(\Delta_\infty)$, the two functions $f \circ \Phi, g \circ \Phi$ are in $B(E)$. From the reversibility of $\mathbf{X}(t)$, we have for any $t > 0$,

$$\begin{aligned} \int_{\Delta_\infty} f(\mathbf{y}) E_{\mathbf{y}}[g(\mathbf{y}(t))] \Xi_{\mathbf{a}, \mathbf{b}}(d\mathbf{y}) &= \int_E f(\Phi(\mathbf{x})) E_{\mathbf{x}}[g(\Phi(\mathbf{x}(t)))] \mu_{\mathbf{a}, \mathbf{b}}(d\mathbf{x}) \\ &= \int_E g(\Phi(\mathbf{x})) E_{\mathbf{x}}[f(\Phi(\mathbf{x}(t)))] \mu_{\mathbf{a}, \mathbf{b}}(d\mathbf{x}) \\ &= \int_{\Delta_\infty} g(\mathbf{y}) E_{\mathbf{y}}[f(\mathbf{y}(t))] \Xi_{\mathbf{a}, \mathbf{b}}(d\mathbf{y}). \end{aligned}$$

Hence $\mathbf{Y}(t)$ is reversible with reversible measure $\Xi_{\mathbf{a}, \mathbf{b}}$. \square

Remark. The one parameter GEM distribution Π_θ^{gem} corresponds to $a_i = \frac{1}{2}, b_i = \frac{\theta}{2}$, and the two parameter GEM distribution $\Pi_{\alpha, \theta}^{gem}$ corresponds to $a_i = \frac{1-\alpha}{2}, b_i = \frac{\theta+i\alpha}{2}$.

3 Uniqueness and Poincaré/Log-Sobolev Inequalities

Let

$$\bar{\Delta}_\infty := \{\mathbf{x} \in [0, 1]^\mathbb{N} : \sum_{i=1}^{\infty} x_i \leq 1\}$$

be the closure of space Δ_∞ in $\mathbb{R}^\mathbb{N}$ under the topology induced by cylindrically continuous functions. The probability $\Xi_{\mathbf{a}, \mathbf{b}}$ can be extended to the space $\bar{\Delta}_\infty$. For simplicity, the same notation is used to denote this extended probability measure.

Now, for $\mathbf{x} \in \bar{\Delta}_\infty$ such that

$$\sum_{i=1}^n x_i < 1, \text{ for all finite } n,$$

let

$$\mathcal{L}(\mathbf{x}) = \sum_{i,j=1}^{\infty} a_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{\infty} b_i(\mathbf{x}) \frac{\partial}{\partial x_i},$$

where

$$\begin{aligned} a_{ij}(\mathbf{x}) &:= x_i x_j \sum_{k=1}^{i \wedge j} \frac{(\delta_{ki}(1 - \sum_{l=1}^{k-1} x_l) - x_k)(\delta_{kj}(1 - \sum_{l=1}^{k-1} x_l) - x_k)}{x_k(1 - \sum_{l=1}^k x_l)}, \\ b_i(\mathbf{x}) &:= x_i \sum_{k=1}^i \frac{(\delta_{ik}(1 - \sum_{l=1}^{k-1} x_l) - x_k)(a_k(1 - \sum_{l=1}^{k-1} x_l) - (a_k + b_k)x_k)}{x_k(1 - \sum_{l=1}^k x_l)}. \end{aligned}$$

Here and in what follows, we set $\sum_{i=1}^0 = 0$ and $\prod_{i=1}^0 = 1$ by conventions. By treating $\frac{0}{0}$ as one, the definition of $\mathcal{L}(\mathbf{x})$ can be extended to all points in $\bar{\Delta}_\infty$. Through direct calculation one can see that \mathcal{L} is the generator of the GEM process.

It follows from direct calculation that

$$(3.1) \quad \sum_{i,j=1}^{\infty} |a_{ij}(x)| \leq 3, \quad |b_i(x)| \leq \sum_{k=1}^i (b_k x_k + a_k), \quad x \in \bar{\Delta}_\infty.$$

Indeed, since $1 - \sum_{l=1}^{i-1} x_l \geq x_i$ and $\sum_{1 \leq i < j < \infty} x_i x_j \leq \frac{1}{2}$, we obtain

$$\begin{aligned}
\sum_{i,j=1}^{\infty} |a_{ij}(x)| &= \sum_{i=1}^{\infty} a_{ii}(x) + 2 \sum_{1 \leq i < j < \infty} |a_{ij}(x)| \\
&\leq \sum_{i=1}^{\infty} x_i^2 \left(\frac{1 - \sum_{l=1}^i x_l}{x_i} + \sum_{k=1}^{i-1} \frac{x_k}{1 - \sum_{l=1}^k x_l} \right) \\
&\quad + 2 \sum_{1 \leq i < j < \infty} x_i x_j \left(1 + \sum_{k=1}^{i-1} \frac{x_k}{1 - \sum_{l=1}^k x_l} \right) \\
&\leq \sum_{i=1}^{\infty} x_i \left(1 - \sum_{l=1}^i x_l + \sum_{k=1}^{i-1} x_k \right) \\
&\quad + 2 \sum_{i=1}^{\infty} x_i \sum_{j=i+1}^{\infty} x_j \left(1 + \frac{\sum_{k=1}^{i-1} x_k}{\sum_{l=i+1}^{\infty} x_l} \right) \\
&\leq 1 + 2 = 3.
\end{aligned}$$

Thus, the first inequality in (3.1) holds. Similarly, the second inequality also holds.

Let

$$\Gamma(f, g)(x) = \sum_{i,j=1}^{\infty} a_{ij}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j}.$$

Then $\Gamma(f, f) \in C_b(\bar{\Delta}_{\infty})$ for any $f \in C_b^1(\bar{\Delta}_{\infty})$.

For each $a > 0, b > 0$, let $\alpha_{a,b}$ be the largest constant such that for $f \in C_b^1([0, 1])$ the log-Sobolev inequality

$$(3.2) \quad \pi_{a,b}(f^2 \log f^2) \leq \frac{1}{\alpha_{a,b}} \int_0^1 x(1-x) f'(x)^2 \pi_{a,b}(dx) + \pi_{a,b}(f^2) \log \pi_{a,b}(f^2)$$

holds. According to [13, Lemma 2.7], we have $\alpha_{a,b} \geq \frac{a \wedge b}{320}$. Moreover, it is easy to see that for $a, b > 0$ the operator

$$r(1-r) \frac{d^2}{dr^2} + (a - (a+b)r) \frac{d}{dr}$$

on $[0, 1]$ has a spectral gap $a+b$ with eigenfunction $h(r) := a - (a+b)r$. So, the Poincaré inequality

$$(3.3) \quad \pi_{a,b}(f^2) \leq \frac{1}{a+b} \int_0^1 x(1-x) f'(x)^2 \pi_{a,b}(dx) + \pi_{a,b}(f)^2$$

holds.

Let $C_{cl}^\infty([0, 1]^\mathbb{N})$ denote the set of all bounded, C^∞ cylindrical functions on $[0, 1]^\mathbb{N}$, and

$$\mathcal{F}C_b^\infty = \{f|_{\bar{\Delta}_\infty} : f \in C_{cl}^\infty([0, 1]^\mathbb{N})\}.$$

Then we have the following theorem.

Theorem 3.1 *For any $f, g \in \mathcal{F}C_b^\infty$, we have*

$$(3.4) \quad \mathcal{E}(f, g) := \Xi_{\mathbf{a}, \mathbf{b}}(\Gamma(f, g)) = -\Xi_{\mathbf{a}, \mathbf{b}}(f \mathcal{L}g).$$

Consequently, $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable in $L^2(\bar{\Delta}_\infty; \Xi_{\mathbf{a}, \mathbf{b}})$ and its closure is a conservative regular Dirichlet form, which satisfies the Poincaré inequality

$$(3.5) \quad \Xi_{\mathbf{a}, \mathbf{b}}(f^2) \leq \frac{1}{\inf_{i \geq 1} (a_i + b_i)} \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}), \Xi_{\mathbf{a}, \mathbf{b}}(f) = 0.$$

If moreover $\inf\{a_i \wedge b_i : i \geq 1\} > 0$, the log-Sobolev inequality

$$(3.6) \quad \Xi_{\mathbf{a}, \mathbf{b}}(f^2 \log f^2) \leq \frac{1}{\beta_{\mathbf{a}, \mathbf{b}}} \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}), \Xi_{\mathbf{a}, \mathbf{b}}(f^2) = 1$$

holds for some $\beta_{\mathbf{a}, \mathbf{b}} \geq \inf\{\frac{a_i \wedge b_i}{320} : i \geq 1\} > 0$.

Proof: For any $f, g \in \mathcal{F}C_b^\infty$, there exists $n \geq 1$ such that

$$(3.7) \quad f(\mathbf{x}) = f(x_1, \dots, x_n), \quad g(\mathbf{x}) = g(x_1, \dots, x_n), \quad \mathbf{x} = (x_1, \dots, x_n, \dots) \in [0, 1]^\mathbb{N}.$$

Let

$$\varphi^{(n)}(\mathbf{x}) = (\varphi_1(\mathbf{x}), \dots, \varphi_n(\mathbf{x})),$$

which maps $[0, 1]^n$ on to $\Delta_n := \{x \in [0, 1]^\mathbb{N} : \sum_{i=1}^n x_i \leq 1\}$. Define

$$L_n := \sum_{i=1}^n x_i(1 - x_i) \frac{\partial}{\partial x_i^2} + \sum_{i=1}^n (a_i - (a_i + b_i)x_i) \frac{\partial}{\partial x_i},$$

and

$$\pi_{\mathbf{a}, \mathbf{b}}^n = \prod_{i=1}^n \pi_{a_i, b_i}, \quad \Xi^n = \pi_{\mathbf{a}, \mathbf{b}}^n \circ \varphi^{(n)-1}.$$

Then, regarding $\{\Xi^n := \pi_{\mathbf{a}, \mathbf{b}}^n \circ \varphi^{(n)-1} : n \geq 1\}$ as probability measures on

$\bar{\Delta}_\infty$, by letting $\Xi^n := \Xi^n(dx_1 \cdots dx_n) \times \delta_0(dx_{n+1}, \dots)$, it converges weakly to $\Xi_{\mathbf{a}, \mathbf{b}}$. Since L_n is symmetric w.r.t. $\pi_{\mathbf{a}, \mathbf{b}}^n$ we have

$$(3.8) \quad \begin{aligned} & \int_{[0,1]^n} \sum_{i=1}^n x_i(1-x_i) \left(\frac{\partial}{\partial x_i} f \circ \varphi^{(n)} \right) \left(\frac{\partial}{\partial x_i} g \circ \varphi^{(n)} \right) d\pi_{\mathbf{a}, \mathbf{b}}^n \\ &= - \int_{[0,1]^n} g \circ \varphi^{(n)} L_n f \circ \varphi^{(n)} d\pi_{\mathbf{a}, \mathbf{b}}^n. \end{aligned}$$

Noting that

$$\varphi_i(\mathbf{x}) = x_i \prod_{l=1}^{i-1} (1-x_l), \quad x_i = \frac{\varphi_i(\mathbf{x})}{1 - \sum_{l=1}^{i-1} \varphi_l(\mathbf{x})}, \quad i \geq 1,$$

we have

$$\frac{df \circ \varphi^{(n)}(\mathbf{x})}{dx_i} = \sum_{j \geq i} \frac{(\delta_{ij} - x_i) \varphi_j(\mathbf{x})}{x_i(1-x_i)} \frac{df}{d\varphi_j} \circ \varphi^{(n)}(\mathbf{x}).$$

Therefore,

$$(3.9) \quad \begin{aligned} & \int_{[0,1]^n} \sum_{i=1}^n x_i(1-x_i) \left(\frac{\partial}{\partial x_i} f \circ \varphi^{(n)} \right) \left(\frac{\partial}{\partial x_i} g \circ \varphi^{(n)} \right) d\pi_{\mathbf{a}, \mathbf{b}}^n \\ &= \int_{[0,1]^n} \Gamma(f, g) \circ \varphi^{(n)} d\pi_{\mathbf{a}, \mathbf{b}}^n = \int_{\Delta_n} \Gamma(f, g) d\Xi^n. \end{aligned}$$

By (3.1) and (3.7), we have $\Gamma(f, g) \in C_b(\bar{\Delta}_\infty)$ so that the weak convergence of Ξ^n to $\Xi_{\mathbf{a}, \mathbf{b}}$ implies

$$(3.10) \quad \lim_{n \rightarrow \infty} \int_{\Delta_n} \Gamma(f, g) d\Xi^n = \int_{\bar{\Delta}_\infty} \Gamma(f, g) d\Xi_{\mathbf{a}, \mathbf{b}}.$$

Similarly, by straightforward calculations we find

$$L_n f \circ \varphi^{(n)}(\mathbf{x}) = (\mathcal{L}f) \circ \varphi^{(n)}(\mathbf{x}).$$

Moreover, (3.1) and (3.7) imply that $g\mathcal{L}f \in C_b(\bar{\Delta}_\infty)$. Thus we arrive at

$$\lim_{n \rightarrow \infty} \int_{\Delta_n} g \circ \varphi^{(n)} L_n f \circ \varphi^{(n)} d\pi_{\mathbf{a}, \mathbf{b}}^n = \int_{\bar{\Delta}_\infty} g\mathcal{L}f d\Xi_{\mathbf{a}, \mathbf{b}}.$$

Therefore, (3.4) follows by combining this with (3.9) and (3.10). This implies the closability of $(\mathcal{E}, \mathcal{F}C_b^\infty)$, while the regularity of its closure follows from the compactness of $\bar{\Delta}_\infty$ under the usual metric

$$\rho(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|.$$

Indeed, it is trivial that $\mathcal{D}(\mathcal{E}) \cap C_0([0, 1]^{\mathbb{N}}) \supset \mathcal{F}C_b^{\infty}$ which is dense in $\mathcal{D}(\mathcal{E})$ under $\mathcal{E}_1^{1/2}$ given by

$$\mathcal{E}_1(f, f) = \mathcal{E}(f, f) + \|f\|_2^2.$$

Moreover, for any $F \in C(\bar{\Delta}_{\infty}) = C_0(\bar{\Delta}_{\infty})$, by its uniform continuity due to the compactness of the space,

$$\bar{\Delta}_{\infty} \mathbf{x} \mapsto F_n(\mathbf{x}) := F(x_1, \dots, x_n, 0, 0, \dots), \quad n \geq 1$$

is a sequence of continuous cylindric functions converging uniformly to F . Since a cylindric continuous function can be uniformly approximated by functions in $\mathcal{F}C_b^{\infty}$ under the uniform norm, it follows that $\mathcal{F}C_b^{\infty}$ is dense in $C_0(\bar{\Delta}_{\infty})$ under the uniform norm. That is, the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular.

Next, the desired Poincaré and log-Sobolev inequalities can be deduced from (3.3) and (3.2) respectively. For simplicity, we only prove the latter. By the additivity property of the log-Sobolev inequality (cf. [5]),

$$\mu^n(h^2 \log h^2) \leq \frac{1}{\beta_{\mathbf{a}, \mathbf{b}}^n} \int_{[0, 1]^n} \sum_{i=1}^n x_i(1 - x_i) \left(\frac{\partial h}{\partial x_i} \right)^2 d\pi_{\mathbf{a}, \mathbf{b}}^n + \mu^n(h^2) \log \pi_{\mathbf{a}, \mathbf{b}}^n(h^2)$$

holds for all $h \in C_b^1([0, 1]^n)$, where

$$\beta_{\mathbf{a}, \mathbf{b}}^n = \inf\{\alpha_{a_i, b_i} : i = 1, \dots, n\}, f^{(n)}(\mathbf{x}) = f(x_1, \dots, x_n, 0, \dots).$$

Combining this with (3.9), for any $f \in \mathcal{D}$, the domain of \mathcal{L} , we have

$$\Xi^n(f^{(n)2} \log f^{(n)2}) \leq \frac{1}{\beta_{\mathbf{a}, \mathbf{b}}^n} \int_{\Delta_n} \Gamma^{(n)}(f, f) d\Xi^n + \Xi^n(f^{(n)2}) \log \Xi^n(f^{(n)2}).$$

Therefore, as explained above, (3.6) for $f \in \mathcal{D}$ follows immediately by letting $n \rightarrow \infty$. Hence, the proof is completed since $\mathcal{D}(\mathcal{E})$ is the closure of \mathcal{D} under $\mathcal{E}_1^{1/2}$. \square

We remark that since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular, according to [6, 9], (L, \mathcal{D}) generates a Hunt process whose semigroup P_t is unique in $L^2(\Xi_{\mathbf{a}, \mathbf{b}})$. Thus the GEM process constructed in section 2 is the unique Feller process generated by \mathcal{L} . Moreover, it is well-known that the log-Sobolev inequality (3.6) implies that P_t converges to $\Xi_{\mathbf{a}, \mathbf{b}}$ exponentially fast in entropy; more precisely (see e.g. [1, Proposition 2.1]),

$$\Xi_{\mathbf{a}, \mathbf{b}}(P_t f \log P_t f) \leq e^{-4\beta_{\mathbf{a}, \mathbf{b}} t} \Xi_{\mathbf{a}, \mathbf{b}}(f \log f), \quad f \geq 0, \Xi_{\mathbf{a}, \mathbf{b}}(f) = 1.$$

Moreover, due to Gross [4], the log-Sobolev inequality is also equivalent to the hypercontractivity of P_t .

Thus, according to Theorem 3.1, we have constructed a diffusion process which converges to its reversible distribution $\Xi_{\mathbf{a}, \mathbf{b}}$ in entropy exponentially fast.

4 Measure-Valued Process

It was shown in Stannat [13] that the log-Sobolev inequality fails to hold for the Fleming-Viot process with parent independent mutation when there are infinite number of types. In this section, we will construct a class of measure-valued processes for which the log-Sobolev inequality holds even when the number of types is infinity.

Let us first consider a measure-valued processes on a Polish space S induced by the above constructed process and a proper Markov process on $S^{\mathbb{N}}$. More precisely, let $X_t := (X_1(t), \dots, X_n(t), \dots)$ be the Markov process on Δ_{∞} associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and $\xi_t := (\xi_1(t), \dots, \xi_n(t), \dots)$ be a Markov process on $S^{\mathbb{N}}$, independent of X_t . We consider the measure-valued process

$$\eta_t := \sum_{i=1}^{\infty} X_i(t) \delta_{\xi_i(t)},$$

where X_i can be viewed as the proportion of the i -th family in the population, and ξ_i its type or label. Then the above process describes the evolution of all (countably many) families on the space S . Let \mathcal{M}_1 be the set of all probability measures on S . Then the state space of this process is

$$\mathcal{M}_0 := \{\gamma \in \mathcal{M}_1 : \text{supp } \gamma \text{ contains at most countably many points}\},$$

which is dense in \mathcal{M}_1 under the weak topology.

Due to Theorem 3.1, if ξ_t converges to its unique invariant probability measure ν on $S^{\mathbb{N}}$, then η_t converges to $\Pi := (\Xi_{\mathbf{a}, \mathbf{b}} \times \nu) \circ \psi^{-1}$ for

$$\psi : \Delta_{\infty} \times S^{\mathbb{N}} \rightarrow \mathcal{M}_0; \quad \psi(\mathbf{x}, \xi) := \sum_{i=1}^{\infty} x_i \delta_{\xi_i}.$$

Unfortunately the process η_t is in general non-Markovian. So we like to modify the construction by using Dirichlet forms.

Let ν be a probability measure on $S^{\mathbb{N}}$ and $(\mathcal{E}_{S^{\mathbb{N}}}, \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}}))$ a conservative symmetric Dirichlet form on $L^2(\nu)$. We then construct the corresponding quadratic form on $L^2(\mathcal{M}_0; \Pi)$ as follows:

$$\begin{aligned} \mathcal{E}_{\mathcal{M}_0}(F, G) &:= \int_{S^{\mathbb{N}}} \mathcal{E}(F_{\xi}, G_{\xi}) \nu(d\xi) + \int_{\Delta_{\infty}} \mathcal{E}_{S^{\mathbb{N}}}(F_{\mathbf{x}}, G_{\mathbf{x}}) \pi_{a, b}(d\mathbf{x}) \\ F, G \in \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}) &:= \{H \in L^2(\Pi) : H_{\mathbf{x}} := H \circ \psi(x, \cdot) \in \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}}) \text{ for } \Xi_{\mathbf{a}, \mathbf{b}}\text{-a.s. } \mathbf{x}, \\ &\quad H_{\xi} := H \circ \psi(\cdot, \xi) \in \mathcal{D}(\mathcal{E}) \text{ for } \nu\text{-a.s. } \xi, \text{ such that } \mathcal{E}_{\mathcal{M}_0}(H, H) < \infty\}. \end{aligned}$$

Since Π has full mass on \mathcal{M}_0 , to make the state space complete one may also consider the above defined form a symmetric form on $L^2(\mathcal{M}_1; \Pi) (= L^2(\mathcal{M}_0; \Pi))$.

Theorem 4.1 *Assume there exists $\alpha > 0$ such that*

$$\nu(f^2 \log f^2) \leq \frac{1}{\alpha} \mathcal{E}_{S^{\mathbb{N}}}(f, f) + \nu(f^2) \log \nu(f^2), \quad f \in \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}})$$

holds, then

$$(4.1) \quad \Pi(F^2 \log F^2) \leq \frac{1}{\alpha \wedge \beta_{\mathbf{a}, \mathbf{b}}} \mathcal{E}_{\mathcal{M}_0}(F, F) + \Pi(F^2) \log \Pi(F^2), \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}).$$

If moreover $\mathcal{D}(\mathcal{E}_{\mathcal{M}_0}) \subset L^2(\mathcal{M}_1; \Pi)$ is dense, then $(\mathcal{E}_{\mathcal{M}_0}, \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}))$ is a conservative Dirichlet form on $L^2(\mathcal{M}_0; \Pi)$ so that the associated Markov semigroup P_t satisfies

$$(4.2) \quad \Pi(P_t F \log P_t F) \leq \Pi(F \log F) e^{-(\beta_{\mathbf{a}, \mathbf{b}} \wedge \alpha)t}, \quad t \geq 0, F \geq 0, \Pi(F) = 1,$$

and $(\mathcal{E}_{\mathcal{M}_0}, \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}))$ is regular provided so is $(\mathcal{E}_{S^{\mathbb{N}}}, \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}}))$ and S is compact.

Proof: Let

$$\begin{aligned} \mathcal{D}(\tilde{\mathcal{E}}) = \{ \tilde{F} \in L^2(\Xi_{\mathbf{a}, \mathbf{b}} \times \nu) : \tilde{F}(x, \cdot) \in \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}}) \text{ for } \Xi_{\mathbf{a}, \mathbf{b}}\text{-a.s. } x, \\ \tilde{F}(\cdot, \xi) \in \mathcal{D}(\mathcal{E}) \text{ for } \nu\text{-a.s. } \xi, \text{ such that } \tilde{\mathcal{E}}(\tilde{F}, \tilde{F}) < \infty \}, \end{aligned}$$

where

$$\tilde{\mathcal{E}}(\tilde{F}, \tilde{G}) := \int_{\Delta_{\infty}} \mathcal{E}_{S^{\mathbb{N}}}(\tilde{F}(\mathbf{x}, \cdot), \tilde{G}(\mathbf{x}, \cdot)) \Xi_{\mathbf{a}, \mathbf{b}}(d\mathbf{x}) + \int_{S^{\mathbb{N}}} \mathcal{E}(\tilde{F}(\cdot, \xi), \tilde{G}(\cdot, \xi)) \nu(d\xi).$$

Then $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ is a symmetric Dirichlet form on $L^2(\Delta_{\infty} \times S^{\mathbb{N}}; \Xi_{\mathbf{a}, \mathbf{b}} \times \nu)$ and (see e.g. [5, Theorem 2.3])

$$(4.3) \quad (\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(\tilde{F}^2 \log \tilde{F}^2) \leq \frac{1}{\beta_{\mathbf{a}, \mathbf{b}} \wedge \alpha} (\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(\tilde{F}^2), \quad \tilde{F} \in \mathcal{D}(\tilde{\mathcal{E}}), (\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(\tilde{F}^2) = 1.$$

Let \tilde{P}_t be the Markov semigroup associated to $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$. Then (4.2) follows from the fact that $\eta_t = \psi(X(t), \xi(t))$ and (4.3) implies (cf. [1, Proposition 2.1])

$$(\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(\tilde{P}_t G \log \tilde{P}_t G) \leq (\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(G \log G) e^{-4(\beta_{\mathbf{a}, \mathbf{b}} \wedge \alpha)t}$$

for all $t \geq 0$ and nonnegative function G with $(\Xi_{\mathbf{a}, \mathbf{b}} \times \nu)(G) = 1$. Since $F \in \mathcal{D}(\mathcal{E}_{\mathcal{M}_0})$ if and only if $F \circ \psi \in \mathcal{D}(\tilde{\mathcal{E}})$, and

$$\mathcal{E}_{\mathcal{M}_0}(F, F) = \tilde{\mathcal{E}}(F \circ \psi, F \circ \psi),$$

(4.1) follows from (4.3). By the same reason and noting that $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{E}))$ is a Dirichlet form, we conclude that $(\mathcal{E}_{\mathcal{M}_1}, \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}))$ is a Dirichlet form provided it is densely defined on $L^2(\mathcal{M}_1; \Pi)$. Finally, if S is compact then so is \mathcal{M}_1 (under the weak topology). Thus, as explained in the proof of Theorem 3.1, for regular $(\mathcal{E}_{S^\mathbb{N}}, \mathcal{D}(\mathcal{E}_{S^\mathbb{N}}))$ the set

$$\{f(\langle \cdot, g_1 \rangle, \dots, \langle \cdot, g_n \rangle) : n \geq 1, f \in C_b^1(\mathbb{R}^n), g_i \in C(S), 1 \leq i \leq n\} \subset C_0(\mathcal{M}_0) \cap \mathcal{D}(\mathcal{E}_{\mathcal{M}_1})$$

is dense both in $C_0(\mathcal{M}_1)$ ($= C(\mathcal{M}_1)$) under the uniform norm and in $\mathcal{D}(\mathcal{E}_{\mathcal{M}_1})$ under the Sobolev norm. \square

Remark. Obviously, we have a similar assertion for the Poincaré inequality: if there exists $\lambda > 0$ such that

$$\nu(f^2) \leq \frac{1}{\lambda} \mathcal{E}_{S^\mathbb{N}}(f, f) + \nu(f)^2, \quad f \in \mathcal{D}(\mathcal{E}_{S^\mathbb{N}})$$

holds, then

$$\Pi(F^2) \leq \frac{1}{\lambda \wedge \inf_{i \geq 1} (a_i + b_i)} \mathcal{E}_{\mathcal{M}_0}(F, F) + \Pi(F)^2, \quad F \in \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}).$$

To see that the above theorem applies to a class of measure-valued processes on S , we present below a concrete condition on $\mathcal{E}_{S^\mathbb{N}}$ such that assertions in Theorem 4.1 apply. In particular, it is the case if $\mathcal{E}_{S^\mathbb{N}}$ is the Dirichlet form of a particle system without interactions.

Proposition 4.2 *Let ν_i be the i -th marginal distribution of ν and for a function g on S let $g^{(i)}(\xi) := g(\xi_i), i \geq 1$. Assume that*

$$\mathcal{S}_0 := \left\{ g \in C_0(S) : g^{(i)} \in \mathcal{D}(\mathcal{E}_{S^\mathbb{N}}), \sup_{i \geq 1} \mathcal{E}_{S^\mathbb{N}}(g^{(i)}, g^{(i)}) < \infty \right\}$$

is dense in $C_0(S)$. Then $(\mathcal{E}_{\mathcal{M}_0}, \mathcal{D}(\mathcal{E}_{\mathcal{M}_0}))$ is a symmetric Dirichlet form.

Proof: Under the assumption and the fact that $C_{cl}^2(\Delta_\infty)$ is dense in $L^2(\mathcal{M}_0; \Pi)$, the set

$$\mathcal{S} := \{f(\langle \cdot, g_1 \rangle, \dots, \langle \cdot, g_n \rangle) : n \geq 1, f \in C_b^1(\mathbb{R}^n), g_i \in \mathcal{S}_0, 1 \leq i \leq n\}$$

is dense in $L^2(\mathcal{M}_0; \Pi)$. Therefore, by Theorem 4.1 it suffices to show that $\mathcal{S} \subset \mathcal{D}(\mathcal{E}_{\mathcal{M}_0})$; that is, for $F := f(\langle \cdot, g_1 \rangle, \dots, \langle \cdot, g_n \rangle) \in \mathcal{S}$, one has $F \circ \psi \in \mathcal{D}(\tilde{\mathcal{E}})$. Let

$$F_m(\mathbf{x}) = F\left(\sum_{i=1}^m x_i g_1(\xi_i), \dots, \sum_{i=1}^m x_i g_n(\xi_i)\right), \quad \mathbf{x} \in \Delta_\infty, \quad m \geq 1.$$

Since for fixed $\xi \in S^{\mathbb{N}}$,

$$\partial_{x_i} F \circ \psi(\cdot, \xi)(\mathbf{x}) = \sum_{k=1}^n \partial_k f g_k(\xi_i), \quad i \geq 1$$

is uniformly bounded, one has $F_m \in \mathcal{D}(\mathcal{E})$ and (3.1) yields

$$\mathcal{E}(F_m, F_m) \leq C$$

for some constant $C > 0$ and all $m \geq 1$ and $\xi \in S^{\mathbb{N}}$. Thus, $F \circ \psi(\cdot, \xi) \in \mathcal{D}(\mathcal{E})$ for each $\xi \in S^{\mathbb{N}}$ and

$$(4.4) \quad \sup_{\xi} \mathcal{E}(F \circ \psi(\cdot, \xi), F \circ \psi(\cdot, \xi)) \leq C.$$

On the other hand, since $g_k \in \mathcal{S}_0$, $1 \leq k \leq n$, noting that for any $\mathbf{x} \in \Delta_{\infty}$

$$|F \circ \psi(\mathbf{x}, \xi) - F \circ \psi(\mathbf{x}, \xi')|^2 \leq \left(\sum_{k=1}^n \|\partial_k f\|_{\infty} \right)^2 \sum_{i=1}^{\infty} x_i |g_k(\xi_i) - g_k(\xi'_i)|^2,$$

we conclude in the spirit of [9, Proposition I-4.10] that $F \circ \psi(\mathbf{x}, \cdot) \in \mathcal{D}(\mathcal{E}_{S^{\mathbb{N}}})$ and

$$\mathcal{E}_{S^{\mathbb{N}}}(F \circ \psi(\mathbf{x}, \cdot), F \circ \psi(\mathbf{x}, \cdot)) \leq C'$$

for some $C' > 0$ independent of \mathbf{x} . Combining this with (4.4) we obtain $F \circ \psi \in \mathcal{D}(\tilde{\mathcal{E}})$. \square

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References

- [1] BAKRY, D. (1997). *On Sobolev and logarithmic Sobolev inequalities for Markov semigroups*, “New Trends in Stochastic Analysis” (Editors: K. D. Elworthy, S. Kusuoka, I. Shigekawa), Singapore: World Scientific.
- [2] ETHIER, S. N. AND KURTZ, T. G. (1981). The infinitely-many-neutral-alleles diffusion model. *Adv. Appl. Prob.*, **13**, 429-452.
- [3] FENG, S. AND HOPPE, F. M. (1998). Large deviation principles for some random combinatorial structures in population genetics and Brownian motion. *Ann. Appl. Probab.*, Vol. 8, No. 4, 975-994.
- [4] GROSS, L. (1976). Logarithmic Sobolev inequalities. *Amer. J. Math.*, **97**, 1061-1083.
- [5] GROSS, L. (1993). *Logarithmic Sobolev inequalities and contractivity properties of semigroups*, Lecture Notes in Math. 1563, Springer-Verlag.
- [6] FUKUSHIMA, M., OSHIMA, Y. AND TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter.

- [7] HOPPE, F. M. (1984). Pólya-like urns and the Ewens sampling formula. *Journal of Mathematical Biology*, **20**, 91–94.
- [8] KINGMAN, J. C. F. (1975). Random discrete distributions. *J. Roy. Statist. Soc. B*, **37**, 1–22.
- [9] MA, Z. M. AND RÖCKNER, M. (1992). *An introduction to the theory of (non-symmetric) Dirichlet forms*, Berlin: Springer.
- [10] PITMAN, J. (1996). Random discrete distributions invariant under size-biased permutation. *Adv. Appl. Probab.*, **28**, 525–539.
- [11] PITMAN, J. (1995). Exchangeable and partially exchangeable random partitions. *Prob. Theory Rel. Fields*, **102**, 145–158.
- [12] PITMAN, J. AND YOR, M. (1997). The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, Vol. 25, No. 2, 855–900.
- [13] STANNAT, W. (2000), On validity of the log-Sobolev inequality for symmetric Fleming-Viot operators. *Ann. Probab.*, Vol 28, No. 2, 667–684.